π Is Wrong!

Bob Palais

The Opinion column offers mathematicians the opportunity to write about any issue of interest to the international mathematical community. Disagreement and controversy are welcome. The views and opinions expressed here, however, are exclusively those of the author, and neither the publisher nor the editor-in-chief endorses or accepts responsibility for them. An Opinion should be submitted to the editor-in-chief, Chandler Davis.

I know it will be called blasphemy by some, but I believe that π is wrong. For centuries π has received unlimited praise; mathematicians have waxed rhapsodic about its mysteries, used it as a symbol for mathematics societies and mathematics in general, and built it into calculators and programming languages. Even a movie has been named after it.* I am not questioning its irrationality, transcendence, or numerical calculation, but the choice of the number on which we bestow a symbol conveying deep geometric significance. The proper value, which does deserve all of the reverence and adulation bestowed upon the current impostor, is the number now unfortunately known as 2π.

I do not necessarily feel that π can or even should be changed or replaced with an alternative (though I've by now received some good suggestions!), but it is worthwhile to recognize the repercussions of the error as a warning and a lesson in choosing good notational conventions to communicate mathematical ideas. I compare the problem to what would have occurred if Leonhard Euler had defined e to be .3678 . . . (the natural decay factor equal to $\frac{1}{2.718\ldots}$), in which case there would be just as many unfortunate minus signs running around from that choice as there are factors of 2 from $\pi = 3.14\ldots$

The most significant consequence of the misdefinition of π is for early geometry and trigonometry students who are told by mathematicians that radian measure is more natural than degree measure. In a sense it is, since a quarter of a circle is more naturally measured by 1.57 . . . than by 90. Unfortunately, this beautiful idea is sabotaged by the fact that π isn't 6.28 . . . , which would make a quarter of a circle or a *quad*rant equal to a quarter of π radians; a third of a circle, a third of π radians, and so on. The opportunity to impress students with a beautiful and natural simplification is turned into an absurd exercise in memorization and dogma. An enlightening analogy is to leave clocks the way they are but define an hour to be 30 minutes. In that case, 15 minutes or a quarter of a clock would indeed be called half an hour, just as a quarter of a circle is half of π in mathematics! Even mathematically sophisticated software packages prefer to use 90° to indicate a quarter-circle rotation. We can't really blame them for the fact that π is wrong.

Perhaps more convincing to mathematicians is the litany of important theorems and formulas into which this ubiquitous factor of 2 has crept and propagated: Cauchy's integral formula and Fourier series formulas all begin with $\frac{1}{2\pi}$, Stirling's approximation and the Gaussian normal distribution both carry it, the Gauss-Bonnet and Picard theorems have the mark of 2π. (Archimedes showed that the area of the unit sphere is the area of the cylinder of the same radius and height, or twice the circumference of the unit circle: $4\pi = 2(2\pi)$.) The blight of factors of 2 even affects physics, for example in Maxwell's equations (Gauss's law, Ampère's law, Coulomb's constant) and Planck's constant $\frac{h}{2\pi}$. Euler's formula *should* be $e^{i\pi} = 1$ (or $e^{i\pi/2} = -1$, in which case it involves one more fundamental constant, 2, than before). Wouldn't it be nicer if the periods of the fundamental circular functions cos and sin were π rather than 2π? Wouldn't it be nicer if half-plane integrals such as the Hilbert transform were indicated by the *appearance* of a factor of 2 rather than its disappearance?

The sum of the interior angles of a triangle is π, granted. But the sum of the *exterior* angles of *any* polygon,

*For a non-technical movie, the mathematics was surprisingly good, except for the throwaway question "Surely you've tried all of the 216-digit numbers?" At one number per nanosecond, checking all 30-digit numbers would take longer than the life of the universe!

$$\sin(x + \pi\!\!\!\pi) = \sin(x)$$

$$e^{i\pi\!\!\!\pi} = 1$$

$$n! \sim \sqrt{\pi\!\!\!\pi n}\, n^n e^{-n}$$

$$A = \frac{1}{2}\pi\!\!\!\pi r^2$$

$$\hbar = \frac{h}{\pi\!\!\!\pi}$$

$$T = \frac{\pi\!\!\!\pi}{\omega}$$

$$90° = \frac{1}{4}\,\pi\!\!\!\pi \text{ radians- a } quad\text{rant}$$

$$c_n = \frac{1}{\pi\!\!\!\pi}\int_0^{\pi\!\!\!\pi} f(x)e^{inx}\,dx$$

$$f(a) = \frac{1}{\pi\!\!\!\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$\frac{1}{\pi\!\!\!\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = 1$$

$$\text{The } n\text{th roots of unity: } e^{\frac{j\pi\!\!\!\pi i}{n}}, j = 0, \ldots, n-1$$

The author's father, thinking that the formulas with $\pi = 6.28$ looked more wrong than simpler, constructed a macro in which he combined two pi's with their adjacent "legs" tied together as in a three-legged race! The formulas above use that symbol.

from which the sum of the interior angles can easily be derived, and which generalizes to the integral of the curvature of a simple closed curve, is 2π. The natural formula for area of a circle, $\frac{1}{2}\pi r^2$, has the familiar ring of $\frac{1}{2}gt^2$ or $\frac{1}{2}mv^2$; it would have instilled good habits for representing quadratic quantities and foreshadowed the connection between the area of a circle and the integral of circumference (with respect to *radius*) better than πr^2. Another way of putting it is that radius is far more convenient than diameter—consider what the unit circle means. If it weren't, I would agree that the traditional choice of π was right.

Of course you may say that none of this really matters or affects the mathematics, because we may define things however we like; and that is correct. But the analogy with e mentioned above, or the idea of redefining the symbol i to mean $\frac{\sqrt{-1}}{2}$ shows the true folly of π. Neither of these changes would change the mathematics, but nor would anyone deny they are absurd.

What really worries me is that the first thing we broadcast to the cosmos to demonstrate our "intelligence," is 3.14. . . . I am a bit concerned about what the lifeforms who receive it will do after they stop laughing at creatures who must rarely question orthodoxy. Since it is transmitted in binary, we can hope that they overlook what becomes merely a bit shift!

I would not be surprised and would be interested to hear if this idea has been discussed previously, but I was unable to find any reference either in the wonderful *Pi: A Source Book* by Lennart Berggren, Jonathan Borwein, and Peter Borwein, or in Petr Beckmann's *A History of Pi*, or on the Internet. When I have suggested to people that π has a flaw, their reactions range from surprise, amusement, and agreement, to "Of course, I knew it all along," to dismissal, to indignation.

The history† (I was surprised, along with everyone I tell, that the symbol was not in use in ancient Greece): Oughtred used the symbol π/δ in 1647 for the ratio of the periphery of a circle to its diameter. David Gregory (1697) used π/ρ for the ratio of the periphery of a circle to its radius. The first to use π as we use it now was a Welsh mathematician, William Jones, in 1706 when he stated 3.14159 &c. = π. Euler, who had until then been using the letters p and c, adopted the symbol in 1737, leading to its universal acceptance. If only he or Jones had set Gregory's ρ to be 1 instead of Oughtred's δ, our formulas today would be much more elegant and clear.

Acknowledgments

Many thanks to James Tucker, Nelson Beebe, Bill Bynum, Wayne Burleson, Micah Goodman, and Carolyn Connell for their contributions to this paper.

Robert Palais
21485 Wyoming Street
Salt Lake City, UT 84109
USA
e-mail: palais@math.utah.edu

†*Pi: A Source Book*, L. Berggren, J. Borwein, P. Borwein. Springer-Verlag, New York 2000, p. 292.

$$1\pi\!\!\!\pi = 2\pi = 6.283\ldots \text{ is called } \textbf{One turn.}$$

So instead of 90°, the angle of a quadrant and a quarter of an hour being $\frac{\pi}{2}$ ('Pi over two'), it becomes $\frac{1}{4}\,\pi\!\!\!\pi$, or quite naturally, 'A quarter turn'!

Many other formulas simplify:

$$\cos(x+\pi\!\!\!\pi) = \cos(x) \quad \sin(x+\pi\!\!\!\pi) = \sin(x) \qquad \textit{Cos, Sin Periods}$$

$$e^{i\pi\!\!\!\pi} = 1 \qquad \textit{Euler's Formula}$$

$$n! \sim \sqrt{\pi\!\!\!\pi n}\, n^n e^{-n} \qquad \textit{Stirling's Formula}$$

$$A = \frac{1}{2}\pi\!\!\!\pi r^2 \qquad \textit{Area, } (\tfrac{1}{2}mv^2, \tfrac{1}{2}gt^2)$$

$$\hbar = \frac{h}{\pi\!\!\!\pi} \qquad \textit{Dirac's Constant}$$

$$T = \frac{\pi\!\!\!\pi}{\omega} \qquad \textit{Angular Frequency}$$

$$c_n = \frac{1}{\pi\!\!\!\pi}\int_0^{\pi\!\!\!\pi} f(x)e^{inx}\,dx \qquad \textit{Fourier Coefficients}$$

$$f(a) = \frac{1}{\pi\!\!\!\pi i}\int_C \frac{f(z)}{z-a}\,dz \qquad \textit{Cauchy's Formula}$$

$$\frac{1}{\sqrt{\pi\!\!\!\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = 1 \qquad \textit{Gaussian Distribution}$$

$$e^{\frac{j\pi\!\!\!\pi i}{N}}, j = 0, \ldots, n-1 \qquad \textit{Nth Roots Of Unity}$$