Radian Measurement:

What It Is, and How to Calculate It More Easily Using τ Instead of π

by

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Radian measurement

   In trigonometry, pre-calculus, and higher mathematics, angles are usually measured not in degrees but in radians. Why? To some extent the answer to that question requires studying first-semester calculus. Even at the level of trigonometry and pre-calculus, however, we can partially answer the question. Using radians offers a more efficient system for measuring angles involving more than one rotation. Quick, how many degrees does an angle have that spins around eight-and-a-half times? By using radians, we shall see how speedily you can find the answer.

   So then, what is a radian? Mathematically, a radian is defined by the following formula:

$$\theta \equiv \frac{s}{r} \text{ rad}, \text{ where the symbol} \equiv \text{means "is defined as"} \tag{1.01}$$

   In this formula, θ represents the angle being measured, s represents arc length, r represents radius, and rad means radians. Look at the following diagram:

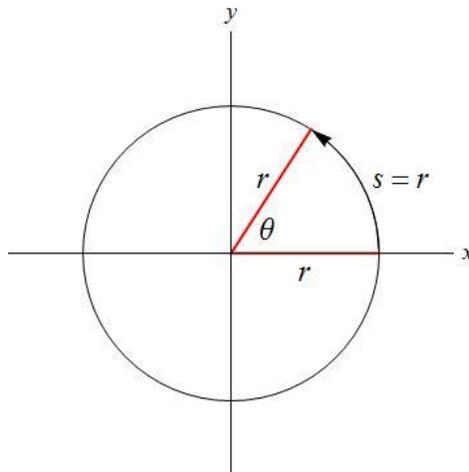**Figure 1**: Arc length (s) = radius (r) when θ = 1 rad. Graphic created using *Mathematica*, v. 8.

   This picture shows that the angle θ is measured as the ratio of the arc length (s) to the radius (r) of an imaginary circle, and if the angular measurement is made in that fashion, then the measurement is made in radians. Now suppose that s equals r. In that case, θ equals one radian. In other words, one radian is the measurement of an angle that is opened just wide enough for the following to happen: If the angle is located at the center of a circle, then the length of the arc that the angle intercepts on the circumference will exactly equal the length of the radius of the circle. Shortly we will see that, converted to degrees, one radian equals just slightly less than $57.3°$.

This raises an interesting question. How many radians can the central angle be? Let's say that the θ is opened up so that the arc length s is twice the radius length. Or three times, or four. How many radians would it take to go around the entire circumference of the circle — no more, no less?

Here is how we answer that last question. If the arc length s goes completely around the circle, then s becomes the circumference C. That is,

$$s = C \tag{1.02}$$

But C is defined in terms of π, as such:

$$\pi \equiv \frac{C}{d} \tag{1.03}$$

This last equation means that π is defined to be the ratio of the circumference of a circle to its diameter.

Of course, $d = 2r$, so we can rewrite Eq. (1.03) in the following way:

$$\pi \equiv \frac{C}{2r}, \tag{1.04}$$

which in turn can be solved for C, as such:

$$C = 2\pi r \tag{1.05}$$

So now we can rewrite $\theta \equiv \frac{s}{r}$ rad to read as follows:

$$360° = \frac{2\pi r}{r} \text{ rad} \tag{1.06}$$

$$360° = 2\pi \text{ rad} \tag{1.07}$$

$$180° = \pi \text{ rad} \tag{1.08}$$

How to convert radians into degrees, and vice versa

Eq. (1.08) is very important, because we use it to convert radian measurement to degree measurement, and vice versa. To go from radians to degrees, we divide both sides of Eq. (1.08) by π to obtain the following:

$$1 \text{ rad } = \frac{180°}{\pi} \tag{1.09}$$

And to convert from degrees to radians, we divide both sides of Eq. (1.08) by 180 to obtain the following:

$$1° = \frac{\pi}{180} \text{ rad} \tag{1.10}$$

Incidentally, pay attention to this convention. If we are measuring angles in degrees, then we must use the degree symbol $(°)$ after the number. If we are measuring angles in radians, however, then we may write rad after the number or we may leave it out. Thus, for example, we may say that $30° = \frac{\pi}{6}$ rad or we may just say that $30° = \frac{\pi}{6}$.

Using Eq. (1.09) and Eq. (1.10), we can generate circles showing special angles, and we can measure these angles using either degrees, like here:
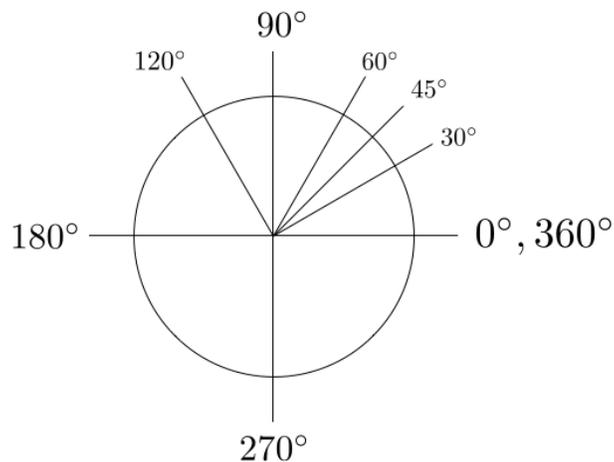

Figure 2: Some special angles, in degrees. Courtesy Michael Hartl, *The Tau Manifesto* (2011). In http://tauday.com.

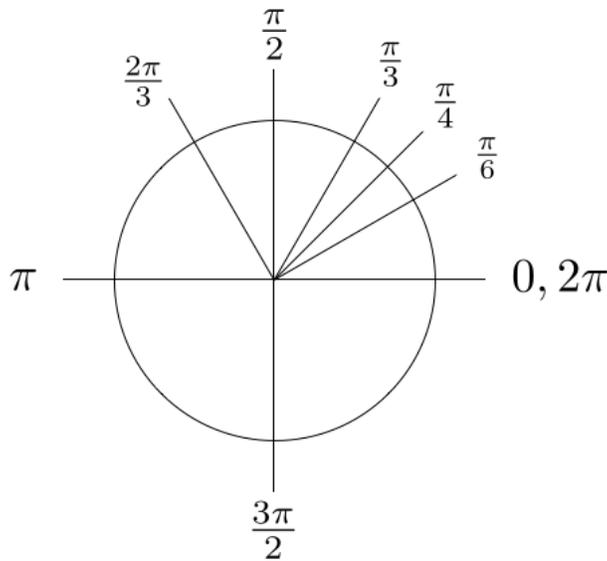Or we can measure the special angles using radian measurement, like this:

A new way of measuring radians: Using τ instead of π

Within the last year or so, a revolutionary new method of thinking about radians has come to the fore. According to this viewpoint, the fundamental circle constant should not be considered in terms of diameter, but rather in terms of radius.

Recall Eq. (1.3) from above:

$$\pi \equiv \frac{C}{d} \tag{1.03}$$

Instead of this definition, let's use the following one instead:

$$\tau \equiv \frac{C}{r} \tag{1.11}$$

Here, small Greek letter *tau* is defined to equal the circumference divided by the radius, and not the circumference divided by the diameter. In addition to thinking of Greek letter *tau* as τ, you can also think of it as a *turn*, meaning a complete rotation of the circle.

Let us restate the equations by which we developed radian measurement in terms of π, and place those equations side by side with formulas to develop radian measurement in terms of τ.

$$\pi \equiv \frac{C}{2r} \qquad (1.04)$$

$$C = 2\pi r \qquad (1.05)$$

$$\theta \equiv \frac{s}{r} \ \text{rad} \qquad (1.01)$$

$$360° = \frac{2\pi r}{r} \ \text{rad} \qquad (1.06)$$

$$360° = 2\pi \ \text{rad} \qquad (1.07)$$

$$180° = \pi \ \text{rad} \qquad (1.08)$$

$$1 \ \text{rad} = \frac{180°}{\pi} \qquad (1.09)$$

$$1° = \frac{\pi}{180} \ \text{rad} \qquad (1.10)$$

$$\tau \equiv \frac{C}{r} \qquad (1.11)$$

$$C = \tau r \qquad (1.12)$$

$$\theta \equiv \frac{s}{r} \ \text{rad} \qquad (1.01)$$

$$360° = \frac{\tau r}{r} \ \text{rad} \qquad (1.13)$$

$$360° = \tau \ \text{rad} \qquad (1.14)$$

$$1 \ \text{rad} = \frac{360°}{\tau} \qquad (1.15)$$

$$1° = \frac{\tau}{360} \ \text{rad} \qquad (1.16)$$

Thus, using τ — and thinking about it as a *turn*, a complete rotation — makes radian measurement more intuitive and therefore easier. One degree is one 360th of a rotation — which makes sense — instead of 180th of π — which has no intuitive appeal. Furthermore, using Eq. (1.15), we can generate a circle showing special angles measured τ-radians.
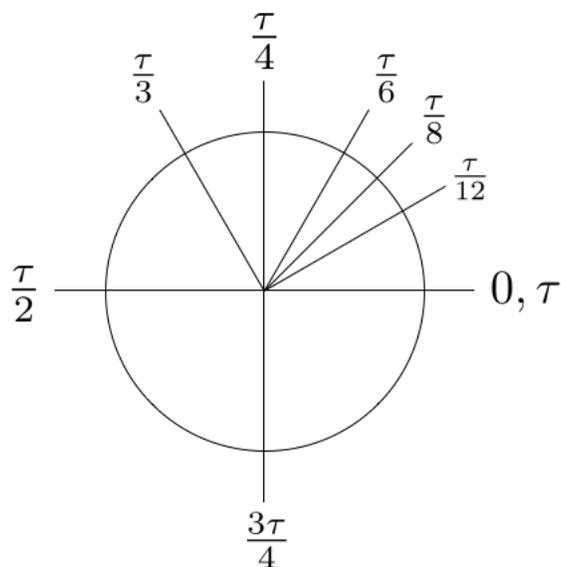
$$\frac{\tau}{4}$$

$$\frac{\tau}{3} \qquad \frac{\tau}{6}$$

$$\frac{\tau}{8}$$

$$\frac{\tau}{12}$$

$$\frac{\tau}{2} \qquad\qquad\qquad 0, \tau$$

$$\frac{3\tau}{4}$$

<u>Figure 4</u>: Some special angles, in τ-radians. Courtesy Michael Hartl, *The Tau Manifesto* (2011). In http://tauday.com.

Using this method, all radian measurements make common sense. For example, one quarter of a turn $(90°)$ equals $\frac{\tau}{4}$ radians — instead of $\frac{\pi}{2}$ radians — and one-eighth of a rotation $(45°)$ equals $\frac{\tau}{8}$ radians — instead of $\frac{\pi}{4}$ radians — and so forth. It could not be simpler and more direct. In fact, we can see that fractions of a circle amount to exactly the same thing as τ-radians.
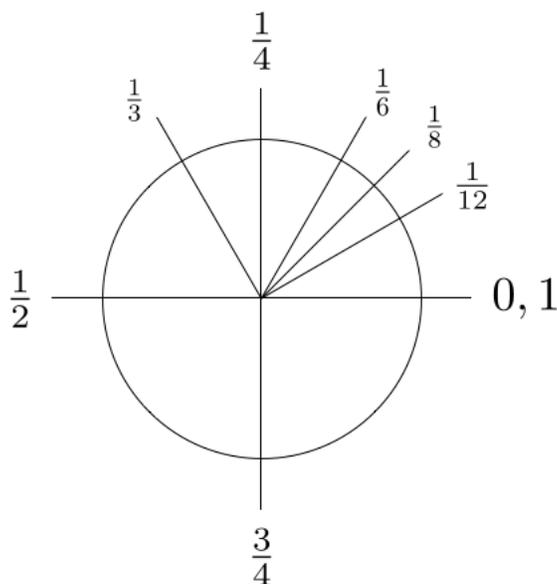
Figure 5: Fractions of a circle are equivalent to τ-radians. Courtesy Michael Hartl, _The Tau Manifesto_ (2011). In http://tauday.com.

How to put this new method into practice

Of course, the conventional method continues to use π-radians, and not τ-radians. Unquestionably, I believe, the force of history causes the problem. For perhaps thousands of years, π, which is defined in terms of diameter, has been used as the fundamental circle constant. And the definition that $180° = \pi$ rad has been used since 1873. If we could just start from scratch, then the logic of using τ, which is defined in terms of radius, would surely be used as the fundamental circle constant instead of the diameter-defined π. In that case, textbooks would be writing that $360° = \tau$ rad, and that would make things a whole lot simpler.

All of this may be well and good. But the influence of history persists, so what should students do when confronting the inefficiency of π-radians? Answer: Either disregard everything that I have just written in this paper and keep using the formulation that $180° = \pi$ rad. Or, better yet, continue to use the conventional definition of radians in terms of π — since that is what all the textbooks do — but when you start getting confused and want to simplify things, just divide by 2. Once you have divided by 2, π-radians becomes τ-radians. So, for example, $180° = \pi$ rad $= \dfrac{\tau}{2}$ rad — one-half of a turn. The conversion is that simple.

Using radians to measure angles involving more than one rotation

At the very beginning of this essay, we claimed that using radians made it easier to measure angles that involved more than one rotation. We used the example of an angle that spins around eight-and-a-half times. Using degrees, the answer involves a clumsy multiplication of 8½ times $360°$ — $3060°$. Using π, the answer is 17 π-radians — much simpler. But simplest of all is using τ — an angle that spins around eight-and-a-half times equals 8½ τ-radians.

Bibliographical essay

In the fall of 2011, I came upon a short article entitled "π Is Wrong!," by Bob Palais of the University of Utah. Prior to reading this article, I was not aware that a controversy even existed regarding whether or not π should be used as the fundamental circle constant, and thus whether or not it should be used to measure radians. I next discovered Michael Hartl's "Tau Manifesto," posted in his website http://tauday.com. Hartl's paper (from which I have used several graphics in this writing) is rather longer than Palais', although Hartl himself acknowledged Palais as a trailblazer in advancing τ. Hartl's article also provides a useful bibliographical essay. The website *Spiked Math* has opened up an enlightening forum on the debate. Furthermore, *Wikipedia*'s entry provides a valuable background to the discussion, including a reference to Joseph Lindenberg. The latter seems to have first come up with the idea in 1988 of using τ rather than π as the fundamental circle constant, if I am correctly reading his instructive post in "Tau Before It Was Cool." Lindenberg also contributed to the forum in *Spiked Math*.

I learned from Russ Rowlett's always useful "How Many? A Dictionary of Units of Measurement" that James Thomson (brother of William Thomson, Lord Kelvin) named and defined the radian in 1873.

Incidentally, from reading contributors' comments in *Wikipedia*, Lindenberg seems to question whether the term "circle constant" has ever been used in any context outside of the π-versus-τ controversy. Apparently, however, he does not object to the term, and he adopts it in his own discussion. I find the term handy, and more to the point I myself have become a staunch "Tauist."

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